Analytical Solution for the Deformation of a Cylinder under Tidal Gravitational Forces

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Abstract. Quite a few future high precision space missions for testing Special and General Relativity will use optical resonators which are used for laser frequency stabilization. These devices are used for carrying out tests of the isotropy of light (Michelson-Morley experiment) and of the universality of the gravitational redshift. As the resonator frequency not only depends on the speed of light but also on the resonator length, the quality of these measurements is very sensitive to elastic deformations of the optical resonator itself. As a consequence, a detailed knowledge about the deformations of the cavity is necessary. Therefore in this article we investigate the modeling of optical resonators in a space environment. Usually for simulation issues the Finite Element Method (FEM) is applied in order to investigate the influence of disturbances on the resonator measurements. However, for a careful control of the numerical quality of FEM simulations a comparison with an analytical solution of a simplified resonator model is beneficial. In this article we present an analytical solution for the problem of an elastic, isotropic, homogeneous free-flying cylinder in space under the influence of a tidal gravitational force. The solution is gained by solving the linear equations of elasticity for special boundary conditions. The applicability of using FEM codes for these simulations shall be verified through the comparison of the analytical solution with the results gained within the FEM code.

1. Motivation

Special (SR) and General Relativity (GR) are two of the most important theories and theoretical frames of modern physics. They are the basis for the understanding of space and time and thus for the underlying physical structure of any other theory. The interest in testing the fundamentals of SR and GR has grown enormously over the last years as all presently discussed approaches to quantum gravity predict tiny violations of SR and GR.

The technological improvements of the last decades have provided scientists with high precision measurement equipment such as optical resonators. In optical resonators laser locking is used to define stable optical frequencies. The resonance frequency of the locked lasers is given by $\nu = mc/L$ where c is the speed of light, L the resonator length L, and m the mode number.

Optical resonators have been used recently to test one of the pillars of Special Relativity, namely the isotropy of the speed of light [7, 8] as well as of the universality of the gravitational redshift [5]. In doing so, two laser beams are locked to two orthogonally oriented resonators. An anisotropic speed of light would lead to a beat of the frequencies during a rotation of this setup. Due to the importance of this type of experiments one looks for ways to improve that. One option for this is to carry out these experiments in space, as planned in the OPTIS mission [19] or with SUMO on the ISS [13].

Although many of the disturbances acting on a resonator can be minimized by means of an appropriate satellite control system, some intrinsic disturbances cannot be eliminated as a matter of principle and distort the resonator shape leading to a systematic frequency shift. In particular the tidal gravitational force ‡ which acts through every extended body cannot be eliminated by choosing an appropriate frame and, thus, will induce distortions on the resonator.

We give a rough estimate of the expected effect of the tidal gravitational force on a freely moving cube of length L. If the position of the cube is at a distance R from the center of the Earth, then the difference of the Earth's acceleration on the top and bottom of the cube is $\Delta a = (\partial^2 U/\partial r^2)L$, where U is the Earth's Newtonian potential $U = GM_\oplus/R$. For an orbit with $R = 10000$ km and a typical resonator length of $L = 5$ cm we have $\Delta a \approx GM_\oplus/R^3 L \approx 2 \cdot 10^{-8}$ m/s^2. In a rough estimate we assume this Δa to act on the top surface of the cube. Now Hook's simple law of elasticity

$$\frac{F}{A} = E\frac{\Delta L}{L} \tag{1}$$

gives the change of the length ΔL of the cube due to a force F acting on the area A. In our case taking $F = m\Delta a = \rho L^3 \Delta a$ we get

$$\frac{\Delta L}{L} = \frac{\rho L}{E}\Delta a \approx 10^{-17} \tag{2}$$

assuming an elasticity modulus of $E = 90$ GPa and a density of 2350 kg/m^3 which is typical for Zerodur.

In the OPTIS mission, for example, the science goal for the measurement of the isotropy of the speed of light is better than $\Delta c/c = 10^{-18}$ [19]. This can only be achieved if the resonator has a length stability of $\Delta L/L = 10^{-18}$ [19]. As one can see from our estimates, the tidal gravitational force will lead to systematic deformations

‡ In space and engineering sciences the tidal gravitational force is often referred to as 'gravity gradient'.

which are one order of magnitude larger than the expected accuracy. Therefore the effect has to be investigated carefully by including the tidal gravitational force into the equations of elasticity, calculating the resulting resonator shape, and then subtract the effect.

Although the linear theory of elasticity has a long history, explicit solutions for special problems are purely spread. In textbooks only examples for simple bodies in homogeneous gravitational fields or for thermal expansions can be found (e.g. [14], [15], [18]). However, most of the solutions employ an ansatz which already includes knowledge about the expected solution. To the understanding of the authors, no publications are available dealing with a body under the influence of a tidal gravitational force so far. The reason for this is probably, that this situation applies only to bodies freely flying in space – a situation which was outside the scope of application in elasticity theory so far.

In the present paper we first derive an analytical solution in terms of a series expansion. This result is then confirmed using numerical methods. These calculations are usually done with help of Finite Element Method (FEM) codes. For most engineering purposes FEM codes are fine. However FEM solutions are only numerical approximations whose accuracy depends highly on the number and shape of the elements that have been chosen to mesh the model. In order to confirm the analytical model and to test the numerical calculation, we compare the analytical with the numerical solution. For this comparison we choose a cylinder as most simple geometry of a body adapted to the symmetry of the problem.

Having thus checked the principal applicability of the FEM methods to these kinds of physical situations, this method safely can be used for calculating the deformations of arbitrarily shaped bodies or for the design of devices insensitive to unwanted influences, or for the elimination of the systematics of the measurements in order to ensure the success of highly sensitive experiments.

2. Basic Equations

2.1. Generalities

The problem of an optical resonator flying on a geodetic Earth orbit can be simplified by treating the problem in a body fixed coordinate system. We also consider, for simplicity, the body to be a homogeneous and isotropic cylinder. The only force present is a volume force due to the tidal gravitational force which will be modeled as gradient of a spherically symmetric Earth acceleration field.

In order to calculate the elastic deformations of the cylinder the equations of elasticity have to be solved including the influence of the tidal gravitational force. The boundary conditions for the solution are given through the condition of weightlessness in space.

As a short introduction, some basic equations of the linear theory of elasticity are given [15, 14, 18, 11]. All equations refer to homogeneous isotropic bodies. Within this paper we do not use the notation within the formalism of the Riemannian geometry (e.g. [20]) but the notation used in [15].

In elasticity the general relation between the stress tensor σ_{ij} and the strain tensor ε_{ij} is given by Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{ij} \qquad i,j,k,l = 1,2,3 \tag{3}$$

where C_{ijkl} is the elasticity tensor related to the material under consideration. For homogeneous isotropic materials the elasticity tensor can be written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \,, \tag{4}$$

where λ and μ are the Lamé constants and δ_{mn} is the Kronecker symbol. Thus Hook's law for homogeneous isotropic materials is

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \ . \tag{5}$$

The strain tensor ε has to fulfill the so-called compatibility condition

$$\frac{\partial^2\varepsilon_{il}}{\partial r_j\partial r_k} + \frac{\partial^2\varepsilon_{jk}}{\partial r_i\partial r_l} - \frac{\partial^2\varepsilon_{jl}}{\partial r_i\partial r_k} - \frac{\partial^2\varepsilon_{ik}}{\partial r_j\partial r_l} = 0 \,, \tag{6}$$

where r_n are the components of the position vector. The relations between strain and the displacement ξ_i are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\xi_i}{\partial r_j} + \frac{\partial\xi_j}{\partial r_i}\right) \ . \tag{7}$$

The equilibrium equation of elasticity describes the equilibrium state of a homogeneous isotropic body when a volume force $\vec{K}$ is acting

$$\frac{\partial}{\partial r_j}\sigma_{ij} + K_i = 0 \ . \tag{8}$$

Applying the relations between stress and displacements the equilibrium equation takes the form [15]

$$(\lambda+\mu)\frac{\partial^2}{\partial r_k\partial r_j}\xi_k + \mu\frac{\partial^2}{\partial r_i\partial r_i}\xi_j + K_j = 0 \,, \tag{9}$$

where $\vec{\xi}$ is the displacement vector. This equation can also be written as

$$\Delta\vec{\xi} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\vec{\xi}) + \frac{1}{\mu}\vec{K} = 0 \tag{10}$$

where ν is the Poisson number which lies between 0 and 0.5 for homogeneous isotropic bodies.

For vanishing volume forces $\vec{K} = 0$ Eq. (9) becomes the homogeneous equilibrium equation

$$(\lambda+\mu)\frac{\partial^2}{\partial r_k\partial r_j}\xi_k + \mu\frac{\partial^2}{\partial r_i\partial r_i}\xi_j = 0 \ . \tag{11}$$

The boundary conditions for the solution of the equilibrium equation are either given by the forces p_i acting on the body surfaces

$$\sigma_{ij}n_j = p_i \tag{12}$$

or by initial displacements ξ_{i0} of the surfaces

$$\xi_i(0) = \xi_{i0} \tag{13}$$

where n_j are the normal vectors on the surfaces.

The general solution of Eq. (10) is a superposition of a homogeneous and a particular solution

$$\vec{\xi} = \vec{\xi}^{\mathrm{h}} + \vec{\xi}^{\mathrm{p}} \,. \tag{14}$$

2.2. The symmetries of our problem

Since we have an axial symmetric problem, we use cylindrical coordinates r, φ, z is useful. All displacements and derivatives with respect to φ vanish and the equilibrium equation of elasticity takes the form (see e.g. [15])

$$
\begin{aligned}
0 &= \Delta\xi_r - \frac{\xi_r}{r^2} + \frac{1}{1-2\nu}\frac{\partial}{\partial r}\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right) \\
0 &= \Delta\xi_z + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right) .
\end{aligned} \tag{15}
$$

The Laplace operator acting on a scalar takes the form

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} . \tag{16}
$$

Note that the Laplace operators acting on a vector field $\vec{\xi}$ takes the form (see [12])

$$
\begin{aligned}
\Delta\xi_r &= \frac{\partial^2 \xi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_r}{\partial r} + \frac{\partial^2 \xi_r}{\partial z^2} - \frac{\xi_r}{r^2} \\
\Delta\xi_z &= \frac{\partial^2 \xi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_z}{\partial r} + \frac{\partial^2 \xi_z}{\partial z^2} .
\end{aligned} \tag{17}
$$

The relations between stresses, strains and displacements are

$$
\begin{aligned}
\sigma_{rr} &= \lambda(\varepsilon_{rr} + \varepsilon_{\varphi\varphi} + \varepsilon_{zz}) + 2\mu\varepsilon_{rr} = \lambda\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right) + 2\mu\frac{\partial \xi_r}{\partial r} \\
\sigma_{\varphi\varphi} &= \lambda(\varepsilon_{rr} + \varepsilon_{\varphi\varphi} + \varepsilon_{zz}) + 2\mu\varepsilon_{\varphi\varphi} = \lambda\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right) + 2\mu\frac{\xi_r}{r} \\
\sigma_{zz} &= \lambda(\varepsilon_{rr} + \varepsilon_{\varphi\varphi} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz} = \lambda\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right) + 2\mu\frac{\partial \xi_z}{\partial z} \\
\sigma_{r\varphi} &= 0 \\
\sigma_{\varphi z} &= 0 \\
\sigma_{rz} &= 2\mu\varepsilon_{rz} = \mu\left(\frac{\partial \xi_r}{\partial z} + \frac{\partial \xi_z}{\partial r}\right) .
\end{aligned} \tag{18}
$$

Beside the axial symmetry we also have the following symmetries for reflection at the $z = 0$ plane: $\xi_z(r, -z) = -\xi_z(r, z)$ and $\xi_r(r, -z) = \xi_r(r, z)$.

3. The Problem

In order to solve the problem of a free-flying isotropic homogeneous cylinder in space the equilibrium equation of elasticity (10) has to be solved. The cylinder has radius R and height $2L$. The body coordinates are (r, φ, z) with the origin being at the center–of–mass of the cylinder. The z–axis coincides with the symmetry axis of the cylinder, see Fig. 1.

The only force present is the volume force $\vec{K}$ which is due to the Earth's gravitational potential U,

$$
\Delta\vec{\xi} + \frac{1}{1-2\nu}\nabla(\nabla \cdot \vec{\xi}) - \frac{1}{\mu}\rho\nabla U = 0 . \tag{19}
$$

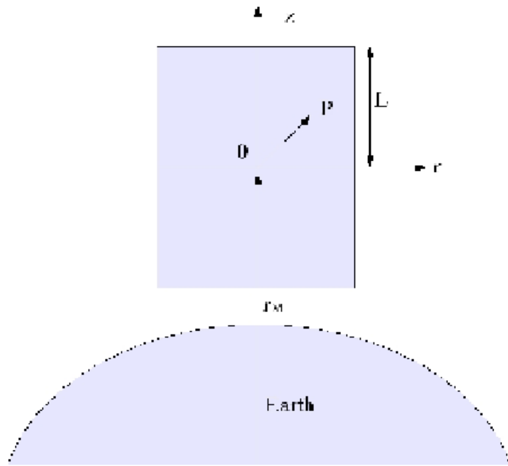

Figure 1. Simplified model of an optical resonator on a geodetic Earth orbit

For a spherical Earth potential, $U(r) = GM_\oplus/r$, where $GM_\oplus$ is the gravitational constant times the mass of Earth, the potential acting at an arbitrary point P inside the cylinder can be calculated via Taylor expansion

$$U(\vec{r}_M + \vec{r}) = U(\vec{r}_M) + \frac{\partial U(\vec{r}_M)}{\partial r_i} r_i + \frac{1}{2}\frac{\partial^2 U(\vec{r}_M)}{\partial r_i \partial r_j} r_i r_j \tag{20}$$

$$= U(\vec{r}_M) + \nabla U(\vec{r}_M)\vec{r} + \frac{GM_\oplus}{2r_M^3}(r^2 - 2z^2) \tag{21}$$

where $\vec{r}_M$ is the vector from the center–of–mass of the Earth to the center–of–mass of the cylinder and $\vec{r}$ is the vector from the cylinder center–of–mass to point P. This Taylor expansion around the center–of–mass of the cylinder to second order gives the axis-symmetric potential in cylindrical coordinates. Note that the linear term of the Taylor expansion vanishes as this equation is valid in the freely falling reference frame of the cylinder.

Since we consider a freely flying cylinder in an orbit around the Earth, no external forces are present and, thus, the forces $\vec{p}$ at the cylinder surfaces are zero which gives us the boundary conditions

$$\sigma_{ij} n_j = p_i = 0 \ . \tag{22}$$

The normal vector $\vec{n} = (n_r\, n_\varphi\, n_z)^T$ (T means the transposed vector) reduces in the axis-symmetric case to $\vec{n} = (n_r\, 0\, n_z)^T$, as the φ component is zero.

Thus the boundary conditions (22) at the top surface of the cylinder, i.e. for $z = L$, $\vec{n} = (0\,0\,1)^T$, are

$$\begin{aligned} p_r(r, z = L) &= 0 = \sigma_{rz}(r, z = L) \\ p_z(r, z = L) &= 0 = \sigma_{zz}(r, z = L) \ . \end{aligned} \tag{23}$$

At the bottom surface of the cylinder, i.e. for $z = -L$, $\vec{n} = (0\,0\,{-1})^T$, we have

$$\begin{aligned} p_r(r, z = -L) &= 0 = -\sigma_{rz}(r, z = -L) \\ p_z(r, z = -L) &= 0 = -\sigma_{zz}(r, z = -L) \ . \end{aligned} \tag{24}$$

Note that these boundary conditions are valid for arbitrary $r \in [0, R)$. For $r = R$ they are not valid as the normal vector is not uniquely defined at the cylinder edges $(r, z) = (R, \pm L)$.

For the superficies cylinder surface, i.e. $r = R$, $\vec{n} = (1\,0\,0)^T$, the boundary conditions are

$$\begin{aligned} p_r(r = R, z) &= 0 = \sigma_{rr}(r = R, z) \\ p_z(r = R, z) &= 0 = \sigma_{zr}(r = R, z) \end{aligned} \tag{25}$$

for all $z \in [0, \pm L)$.

4. The Solution

4.1. Particular Solution

In order to find a particular solution of the problem one can assume that the solution of the equilibrium equation can be written as gradient of a scalar ψ [18]

$$\vec{\xi}^{\mathrm{p}} = \nabla\psi\,. \tag{26}$$

Inserting this approach into Eq. (19) yields

$$\begin{aligned} \nabla\left(\Delta\psi + \frac{1}{1-2\nu}\Delta\psi\right) &= \frac{\rho}{\mu}\nabla U \\ \nabla\left(\frac{2(1-\nu)}{1-2\nu}\Delta\psi\right) &= \frac{\rho}{\mu}\nabla U\ . \end{aligned} \tag{27}$$

Calculation of the volume integral of the divergence gives

$$\Delta\psi = \frac{1-2\nu}{2(1-\nu)\mu}\rho(U + U_0) \tag{28}$$

where U_0 is the integration constant. Inserting the potential U calculated in Eq. (20) and using (16) we obtain

$$\psi = -\frac{1-2\nu}{2(1-\nu)\mu}\rho\frac{GM_\oplus}{2r_M^3}\left(-\frac{r^4}{16} + \frac{1}{6}z^4 + cr^2 + dz^2 + c_1 r + d_1 z + c_2 + d_2\right) \tag{29}$$

where $U_0 = 4c + 2d$. Thus, the displacement vector becomes

$$\vec{\xi}^{\mathrm{p}} = -\frac{1-2\nu}{2(1-\nu)\mu}\rho\frac{GM_\oplus}{2r_M^3}\left(\left(-\frac{r^3}{4} + 2cr + c_1\right)\vec{e_r} + \left(\frac{2}{3}z^3 + 2dz + d_1\right)\vec{e_z}\right)\,. \tag{30}$$

Since the displacement at the center of mass of the cylinder should vanish, $\xi_r(0, 0) = \xi_z(0, 0) = 0$, both, c_1 and d_1, have to be zero. The unknown constants c and d are arbitrary.

The r and z components of the displacement are (with $\mu = \lambda\frac{1-2\nu}{2\nu}$)

$$\begin{aligned} \xi_r^{\mathrm{p}} = &-\frac{1-2\nu}{2(1-\nu)\mu}\rho\frac{GM_\oplus}{2r_M^3}\left(-\frac{r^3}{4} + 2cr\right) = \frac{\nu}{\lambda(1-\nu)}\gamma\left(-\frac{r^3}{4} + 2cr\right) \\ \xi_z^{\mathrm{p}} = &-\frac{1-2\nu}{2(1-\nu)\mu}\rho\frac{GM_\oplus}{2r_M^3}\left(\frac{2}{3}z^3 + 2dz\right) = \frac{\nu}{\lambda(1-\nu)}\gamma\left(\frac{2}{3}z^3 + 2dz\right)\,, \end{aligned} \tag{31}$$

where we substituted $\gamma := -GM/(2r_M^3)\rho$.

Using Eqs. (18) we obtain the stress components out of the displacements

$$\sigma^{\mathrm{p}}_{rr} = \gamma\left(\frac{(2\nu-3)}{4(1-\nu)}r^2 + \frac{2(c+\nu d)}{1-\nu} + \frac{2\nu}{1-\nu}z^2\right)$$
$$\sigma^{\mathrm{p}}_{zz} = \gamma\left(\frac{4c\nu}{1-\nu} - \frac{\nu}{1-\nu}r^2 + 2(z^2+d)\right)$$
$$\sigma^{\mathrm{p}}_{rz} = 0\ . \tag{32}$$

4.2. Homogeneous Solution

4.2.1. Derivation of the boundary conditions A displacement vector which satisfies the homogeneous equation of elasticity

$$\Delta\vec{\xi}^{\mathrm{h}} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\vec{\xi}^{\mathrm{h}}) = 0 \tag{33}$$

has to fulfill the biharmonic equation (see, e.g. [15] and [4])

$$\nabla^2\nabla^2\vec{\xi}^{\mathrm{h}} = \Delta\Delta\vec{\xi}^{\mathrm{h}} = 0\ . \tag{34}$$

Note that the Laplace operator applied to a vector field has the form given in Equation (17).

The boundary conditions the homogeneous solution part has to fulfill can be derived from the boundary conditions (22) of the complete solution. The boundary conditions at the cylinder top surface, i.e. $z = L$, normal vector $\vec{n} = (0\,0\,1)^T$, are

$$p_r(r, z=L) = 0 = \sigma_{rz}(r, z=L) = \sigma^{\mathrm{h}}_{rz}(r, z=L) + \sigma^{\mathrm{p}}_{rz}(r, z=L)$$
$$p_z(r, z=L) = 0 = \sigma_{zz}(r, z=L) = \sigma^{\mathrm{h}}_{zz}(r, z=L) + \sigma^{\mathrm{p}}_{zz}(r, z=L)\ . \tag{35}$$

With help of the expressions of the stress components derived from the particular solution (32) one obtains

$$\sigma^{\mathrm{h}}_{zz}(r, z=L) = -\sigma^{\mathrm{p}}_{zz}(r, z=L) = -\gamma\left(\frac{4c\nu}{1-\nu} - \frac{\nu}{1-\nu}r^2 + 2(L^2+d)\right)$$
$$\sigma^{\mathrm{h}}_{rz}(r, z=L) = -\sigma^{\mathrm{p}}_{rz}(r, z=L) = 0\,. \tag{36}$$

The boundary conditions at the cylinder bottom surface, i.e. $z = -L$, normal vector, $\vec{n} = (0\,0\,{-1})^T$

$$\begin{aligned} p_r(r, z=-L) &= 0 = -\sigma_{rz}(r, z=-L) \\ &= -\sigma^{\mathrm{h}}_{rz}(r, z=-L) - \sigma^{\mathrm{p}}_{rz}(r, z=-L) \\ p_z(r, z=-L) &= 0 = -\sigma_{zz}(r, z=-L) \\ &= -\sigma^{\mathrm{h}}_{zz}(r, z=-L) - \sigma^{\mathrm{p}}_{zz}(r, z=-L) \end{aligned} \tag{37}$$

give the same boundary conditions as for the top surface, as z occurs only as z^2 term

$$\begin{aligned} \sigma^{\mathrm{h}}_{zz}(r, z=-L) &= -\sigma^{\mathrm{p}}_{zz}(r, z=-L) \\ &= -\gamma\left(\frac{4c\nu}{1-\nu} - \frac{\nu}{1-\nu}r^2 + 2(L^2+d)\right) \\ \sigma^{\mathrm{h}}_{rz}(r, z=-L) &= -\sigma^{\mathrm{p}}_{rz}(r, z=-L) = 0 \qquad . \end{aligned} \tag{38}$$

For the superficies surface, i.e. $r = R$, $\vec{n} = (1\,0\,0)^T$

$$p_r(r=R, z) = 0 = \sigma_{rr}(r=R, z) = \sigma^{\mathrm{h}}_{rr}(r=R, z) + \sigma^{\mathrm{p}}_{rr}(r=R, z)$$
$$p_z(r=R, z) = 0 = \sigma_{zr}(r=R, z) = \sigma^{\mathrm{h}}_{zr}(r=R, z) + \sigma^{\mathrm{p}}_{zr}(r=R, z) \tag{39}$$

one gets

$$\begin{aligned}
\sigma^{\mathrm{h}}_{rr}(r=R,z) =& -\sigma^{\mathrm{p}}_{rr}(r=R,z) \\
=& -\gamma\left(\frac{(2\nu-3)}{4(1-\nu)}R^2 + \frac{2(c+\nu d)}{1-\nu} + \frac{2\nu}{1-\nu}z^2\right) \\
\sigma^{\mathrm{h}}_{zr}(r=R,z) =& -\sigma^{\mathrm{p}}_{zr}(r=R,z) = 0 \qquad .
\end{aligned} \tag{40}$$

4.2.2. General Ansatz for the Homogeneous Solution Love [17] showed, that the deformations in an elastic axis-symmetric body can be expressed in terms of the so-called Love function χ

$$\begin{aligned}
\xi^{\mathrm{h}}_r =& -\frac{1+\nu}{E}\frac{\partial^2\chi}{\partial r\partial z} \\
\xi^{\mathrm{h}}_z =& \frac{1+\nu}{E}\left((1-2\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r}\right)
\end{aligned} \tag{41}$$

where ξ_r and ξ_z are the displacement components.

The corresponding stress components are

$$\begin{aligned}
\sigma^{\mathrm{h}}_{rr} =& \frac{\partial}{\partial z}\left(\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial r^2}\right) \\
\sigma^{\mathrm{h}}_{rz} =& \frac{\partial}{\partial r}\left((1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right) \\
\sigma^{\mathrm{h}}_{zz} =& \frac{\partial}{\partial z}\left((2-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right) \\
\sigma^{\mathrm{h}}_{\phi\phi} =& \frac{\partial}{\partial r}\left(\nu\nabla^2\chi - \frac{1}{r}\frac{\partial\chi}{\partial r}\right) \quad .
\end{aligned} \tag{42}$$

The Love function χ necessarily fulfills the biharmonic equation

$$\nabla^2\nabla^2\chi = 0 \; . \tag{43}$$

The major obstacle is to find an adequate approach for the Love function fulfilling all boundary conditions. A general approach is the separation of variables, that means a factorization of the kind $\chi(r,z) = F_1(r)\cdot F_2(z)$. For axial symmetric problems the Bessel functions (see Appendix A) are the natural choice to represent the r dependence. Owing to the additional reflection symmetry and antisymmetry of ξ_r and ξ_z, the z dependence can be represented by the trigonometric functions cosine and sine. Bessel functions as well as trigonometric functions form a complete orthogonal set of functions so that each function can be represented as a series of these sets.

For the solution of the homogeneous equation of elasticity a so-called Papkovich-Neuber approach for the displacement field can be made [12]. One writes $\vec{\xi} = \nabla\Phi + 4(1-\nu)\vec{B} - \nabla\left(\vec{B}\cdot\vec{r} + B_0\right)$, where Φ is a scalar, and the vector $\vec{B}$ fulfills the biharmonic equation. For $\vec{B}$ and B_0 the approach of separation of variables is used where the r dependency is represented by Bessel functions and the z dependency is represented by trigonometric functions. By modifying this approach one can make an ansatz for the Love function suggested in [21]

$$\begin{aligned}
\chi = B_0 z^3 + \sum_{j=1}^{\infty}\left(A_j\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j z\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)}\right)\frac{J_0(\lambda_j r)}{\lambda_j^2} \\
+ D_0 r^2 z + \sum_{n=1}^{\infty}\left(C_n\frac{I_0(k_n r)}{I_1(k_n R)} + D_n r\frac{I_1(k_n r)}{I_1(k_n R)}\right)\frac{\sin(k_n z)}{k_n^2} \; .
\end{aligned} \tag{44}$$

Herein J_0 are the Bessel functions of first kind and order zero and I_1 are the modified Bessel functions of order one (see Appendix A). Furthermore $\zeta_j = \lambda_j R$ are the zeros of the Bessel functions of order one, $J_1(\zeta_j) = 0$, and $k_n = \frac{n\pi}{L}$ where n is an integer number.

If we insert the Love function approach into (42) we obtain for the stress components

$$
\begin{aligned}
\sigma^{\mathrm{h}}_{rr} &= 6\nu B_0 + (4\nu - 2)D_0 \\
&+ \sum_{j=1}^{\infty}\left(A_j\lambda_j\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j\left((1+2\nu)\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + \lambda_j z\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)}\right)\right)J_0(\lambda_j r) \\
&- \sum_{j=1}^{\infty}\left(A_j\lambda_j\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j\left(\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + \lambda_j z\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)}\right)\right)\frac{J_1(\lambda_j r)}{\lambda_j r} \\
&- \sum_{n=1}^{\infty}\left(C_n k_n\left(\frac{I_0(k_n r)}{I_1(k_n R)} - \frac{1}{k_n r}\frac{I_1(k_n r)}{I_1(k_n R)}\right)\right. \\
&\left. + D_n\left(k_n r\frac{I_1(k_n r)}{I_1(k_n R)} + (1-2\nu)\frac{I_0(k_n r)}{I_1(k_n R)}\right)\right)\cos(k_n z) \\
\sigma^{\mathrm{h}}_{zz} &= (6-6\nu)B_0 + (8-4\nu)D_0 \\
&- \sum_{j=1}^{\infty}\left(A_j\lambda_j\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j\left((2\nu-1)\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + \lambda_j z\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)}\right)\right)J_0(\lambda_j r) \\
&+ \sum_{n=1}^{\infty}\left(C_n k_n\frac{I_0(k_n r)}{I_1(k_n R)} + D_n\left((4-2\nu)\frac{I_0(k_n r)}{I_1(k_n R)} + k_n r\frac{I_1(k_n r)}{I_1(k_n R)}\right)\right)\cos(k_n z) \\
\sigma^{\mathrm{h}}_{rz} &= \sum_{j=1}^{\infty}\left(A_j\lambda_j\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j\left(2\nu\frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} + \lambda_j z\frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)}\right)\right)J_1(\lambda_j r) \\
&+ \sum_{n=1}^{\infty}\left(C_n k_n\frac{I_1(k_n r)}{I_1(k_n R)} + D_n\left((2-2\nu)\frac{I_1(k_n r)}{I_1(k_n R)} + k_n r\frac{I_0(k_n r)}{I_1(k_n R)}\right)\right)\sin(k_n z)\ .
\end{aligned}
\tag{45}
$$

The appearance of the coefficients A_j and B_j in the Dini and Bessel-Fourier series ($d_0 + \sum_{n=1}^{\infty} d_j J_0(\lambda_j r)$ and $\sum_{n=1}^{\infty} c_j J_1(\lambda_j r)$, see Appendix A) allows us to represent arbitrary boundary conditions for σ_{zz} and σ_{rz} at the cylinder top and bottom surfaces ($z = \pm L$). Similarly, the appearance of the coefficients C_n and D_n in the Fourier series ($a_0/2 + \sum_{n=1}^{\infty} a_n \cos(k_n z) + \sum_{n=1}^{\infty} b_n \sin(k_n z)$, see Appendix B) allows us to describe arbitrary boundary conditions for σ_{rr} and σ_{rz} at the superficies surface of the cylinder. By using the boundary conditions we can now determine the unknown coefficients.

4.2.3. Determination of the Coefficients A_j and C_n For the boundary conditions $0 = \sigma^{\mathrm{h}}_{rz}(r, z = \pm L)$ (Eqs. (36) and (38)) we get

$$
0 = \sum_{j=1}^{\infty}\left(A_j\lambda_j + B_j\left(2\nu + \lambda_j L\coth(\lambda_j L)\right)\right)J_1(\lambda_j r)\ . \tag{46}
$$

Multiplication with $kJ_1(\lambda_{j'} r) r dr$ and subsequent integration over the interval 0 to R yields with help of (A.14)

$$
A_j\lambda_j = -B_j\left(2\nu + \lambda_j L\coth(\lambda_j L)\right) \qquad j = 1 \ldots \infty\ . \tag{47}
$$

For the boundary conditions $0 = \sigma^{\mathrm{h}}_{rz}(r = R, z)$ (Eq. 39) we get

$$0 = \sum_{n=1}^{\infty} \left(C_n k_n + D_n \left((2 - 2\nu) + k_n R \frac{I_0(k_n r)}{I_1(k_n R)} \right) \right) \sin(k_n z) \ . \tag{48}$$

Multiplication with $\sin(k_{n'} z)dz$ and subsequent integration over the interval $-L$ to L yields with help of (B.2)

$$C_n k_n = -D_n \left((2 - 2\nu) + k_n R \frac{I_0(k_n R)}{I_1(k_n R)} \right) \qquad n = 1 \ldots \infty \ . \tag{49}$$

4.2.4. Determination of the Coefficients B_j and D_n A simpler expression for the boundary conditions can be obtained by inserting (47) and (49) into (45),

$$\begin{aligned} \sigma^{\mathrm{h}}_{rr}(r = R, z) &= -\sigma^{\mathrm{p}}_{rr}(r = R, z) \\ &= 6\nu B_0 + (4\nu - 2)D_0 \\ &\quad + \sum_{j=1}^{\infty} B_j \left((1 - \lambda_j L \coth(\lambda_j L)) \frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} + \lambda_j z \frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} \right) J_0(\lambda_j R) \\ &\quad + \sum_{n=1}^{\infty} D_n \left(k_n R \left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2 - 2\nu}{k_n R} \right) \cos(k_n z) \qquad (50) \\ \sigma^{\mathrm{h}}_{zz}(r, z = \pm L) &= -\sigma^{\mathrm{p}}_{zz}(r, z = \pm L) \\ &= (6 - 6\nu)B_0 + (8 - 4\nu)D_0 \\ &\quad + \sum_{j=1}^{\infty} B_j \left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} \right) J_0(\lambda_j r) \\ &\quad + \sum_{n=1}^{\infty} (-1)^n D_n \left(k_n r \frac{I_1(k_n r)}{I_1(k_n R)} + \left(2 - k_n R \frac{I_0(k_n R)}{I_1(k_n R)} \right) \frac{I_0(k_n r)}{I_1(k_n R)} \right) \quad (51) \end{aligned}$$

Again $J_1(\lambda_j R) = 0$ and $\cos(k_n L) = \cos(n\pi) = (-1)^n$ were used.

Expansion of the Particular Boundary Conditions in Dini and Fourier Series As the Bessel functions form a complete orthogonal set of functions, each function can be represented by a Bessel series as given in (A.6). With the additional condition that $\zeta_j = \lambda_j R$ are the zeros of J_1 and $\frac{\partial}{\partial r} J_0(\lambda_j r)|_{r=R} = -\lambda_j J_1(\lambda_j r)|_{r=R} = -\lambda_j J_1(\lambda_j R) = -\lambda_j R J_1(\zeta_j) = 0$, the particular boundary condition part $\sigma^{\mathrm{p}}_{zz}(r, z = \pm L)$ can be represented by a Dini series expansion as given in Equation (A.11)

$$\sigma^{\mathrm{p}}_{zz}(r, z = \pm L) = \sigma^{\mathrm{p}}_{zz}(r) = d_0 + \sum_{j=1}^{\infty} d_j J_0(\lambda_j r) \tag{52}$$

with

$$d_0 = \frac{2}{R^2} \int_0^R \sigma^{\mathrm{p}}_{zz}(r) r dr = \hat{c}_0 - \frac{\hat{c}_2 R^2}{4} \tag{53}$$

where $\hat{c}_0 = \gamma \left(\frac{4c\nu}{1-\nu} + 2(L^2 + d) \right)$ and $\hat{c}_2 = 2\gamma \frac{\nu}{1-\nu}$ were substituted and

$$d_j = \frac{2}{R^2 [J_0(\zeta_j)]^2} \int_0^R \sigma^{\mathrm{p}}_{zz}(r) J_0(\lambda_j r) r dr \tag{54}$$

$$= -\frac{4}{\lambda_j^2 J_0(\zeta_j)} \gamma \frac{\nu}{1 - \nu} \qquad j = 1 \ldots \infty \ . \tag{55}$$

Note that, following (A.16), $\int_0^R rJ_0(\lambda_j r)rdr = 0$ if $\zeta_j = \lambda_j R$ are the zeros of J_1.

Therefore the particular boundary condition $\sigma^{\mathrm{p}}_{zz}(r, z = \pm L)$ can be written as

$$\sigma^{\mathrm{p}}_{zz}(r) = \hat{c}_0 - \frac{\hat{c}_2 R^2}{4} - \sum_{j=1}^{\infty} \frac{2}{\lambda_j^2 J_0(\zeta_j)} \hat{c}_2 J_0(\lambda_j r) \ . \tag{56}$$

Similarly, each function can be represented by a Fourier series, Eq. (B.3)

$$\sigma^{\mathrm{p}}_{rr}(r = R, z) = \sigma^{\mathrm{p}}_{rr}(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n z) + \sum_{n=1}^{\infty} b_n \sin(k_n z) \tag{57}$$

and the Fourier coefficients can be determined from (B.4). This gives a new representation of the particular boundary condition $\sigma^{\mathrm{p}}_{rr}(r = R, z)$

$$\sigma^{\mathrm{p}}_{rr}(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n z) = \hat{c}_1 + \frac{L^2 \hat{c}_2}{3} + \frac{4L^2 \hat{c}_2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(k_n z) \,, \tag{58}$$

where

$$\hat{c}_1 = \gamma \left(\frac{(2\nu - 3)}{4(1-\nu)} R^2 + \frac{2(c + \nu d)}{1-\nu} \right) \qquad \text{and} \qquad \hat{c}_2 = \gamma \frac{2\nu}{1-\nu} \tag{59}$$

with the arbitrary constants c and d from the particular solution part.

Inserting the Particular Boundary Conditions After having found representations for the particular boundary conditions in terms of cos and J_0 we can determine the remaining unknown coefficients from Eqs. (50) and (51). In doing so we first simplify these equations with help of (B.5), (B.6) and (B.7)

$$\begin{aligned} -\sigma^{\mathrm{p}}_{rr}(r = R, z) &= \sigma^{\mathrm{h}}_{rr}(r = R, z) \\ -\hat{c}_1 - \frac{L^2\hat{c}_2}{3} - \frac{4L^2\hat{c}_2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(k_n z) &= 6\nu B_0 + (4\nu - 2) D_0 \\ &\quad + \sum_{j=1}^{\infty} B_j \sum_{n=1}^{\infty} (-1)^n \frac{4\lambda_j k_n^2}{L(k_n^2 + \lambda_j^2)^2} \cos(k_n z) J_0(\lambda_j R) \\ &\quad + \sum_{n=1}^{\infty} D_n \left(k_n R \left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2 - 2\nu}{k_n R} \right) \cos(k_n z) \end{aligned} \tag{60}$$

$$\begin{aligned} -\sigma^{\mathrm{p}}_{zz}(r, z = \pm L) &= \sigma^{\mathrm{h}}_{zz}(r, z = \pm L) \\ -\hat{c}_0 + \frac{\hat{c}_2 R^2}{4} + \sum_{j=1}^{\infty} \frac{2}{\lambda_j^2 J_0(\zeta_j)} \hat{c}_2 J_0(\lambda_j r) &= (6 - 6\nu) B_0 + (8 - 4\nu) D_0 \\ &\quad + \sum_{j=1}^{\infty} B_j \left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} \right) J_0(\lambda_j r) \\ &\quad + \sum_{n=1}^{\infty} (-1)^n D_n k_n \left(\sum_{j=1}^{\infty} \frac{4\lambda_j^2}{R(k_n^2 + \lambda_j^2)^2} \frac{J_0(\lambda_j r)}{J_0(\lambda_j R)} \right) \ . \end{aligned} \tag{61}$$

Multiplication of (60) with $\cos(k_{n'} z)$ and subsequent integration over the interval from $-L$ to L gives with help of (B.2)

$$-\frac{4L^2\hat{c}_2}{\pi^2 n^2} = \sum_{j=1}^{\infty} B_j \frac{4\lambda_j k_n^2}{L(k_n^2 + \lambda_j^2)^2} J_0(\lambda_j R) + (-1)^n D_n \left(k_n R \left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2 - 2\nu}{k_n R} \right) . \tag{62}$$

This results in an equation for the coefficients D_n in dependence of B_n

$$D_n = -\left(k_n R\left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2}-1\right)-\frac{2-2\nu}{k_n R}\right)^{-1}\times$$
$$\times\left(\sum_{m=1}^{\infty} B_m(-1)^n \frac{4\lambda_m k_n^2}{L(k_n^2+\lambda_m^2)^2} J_0(\lambda_m R)+4\frac{(-1)^n \hat{c}_2}{k_n^2}\right) . \tag{63}$$

In the same way multiplication of (61) with $J_0(\lambda_{j'} r)r$ and subsequent integration over the interval from 0 to R gives with help of (A.15)

$$\frac{2}{\lambda_j^2}\hat{c}_2 = J_0(\lambda_j R)B_j\left(\coth(\lambda_j L)+\frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right)$$
$$+\sum_{n=1}^{\infty}(-1)^n D_n k_n\left(\frac{4\lambda_j^2}{R(k_n^2+\lambda_j^2)^2}\right) , \tag{64}$$

what gives an equation for the coefficients B_j in dependence of D_n

$$B_j = \left(J_0(\lambda_j R)\left(\coth(\lambda_j L)+\frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right)\right)^{-1}\times$$
$$\times\left(\frac{2}{\lambda_j^2}\hat{c}_2-\sum_{k=1}^{\infty}(-1)^k D_k k_k\left(\frac{4\lambda_j^2}{R(k_k^2+\lambda_j^2)^2}\right)\right) . \tag{65}$$

4.2.5. Determination of B_0 and D_0 After the determination of the coefficients in the infinite series (44) we determine the remaining unknowns B_0 and D_0. From the boundary condition $\sigma_{rr}^{\mathrm{h}}(r=R,z) = -\sigma_{rr}^{\mathrm{p}}(r=R,z)$, valid for all $z\in[0,\pm L)$, one can deduce that in particular $\sigma_{rr}^{\mathrm{h}}(r=R,z=0) = -\sigma_{rr}^{\mathrm{p}}(r=R,z=0)$ must hold. Then Eq. (50) simplifies to

$$-\hat{c}_1-\frac{L^2\hat{c}_2}{3}-\frac{4L^2\hat{c}_2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 6\nu B_0+(4\nu-2)D_0$$
$$+\sum_{j=1}^{\infty}B_j\left(\frac{(1-\lambda_j L\coth(\lambda_j L))}{\sinh(\lambda_j L)}\right)J_0(\lambda_j R)$$
$$+\sum_{n=1}^{\infty}D_n\left(k_n R\left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2}-1\right)-\frac{2-2\nu}{k_n R}\right) . \tag{66}$$

From $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ we get

$$-\hat{c}_1 = 6\nu B_0+(4\nu-2)D_0+\mathcal{R}\,, \tag{67}$$

where we abbreviated

$$\mathcal{R} = \sum_{j=1}^{\infty}B_j\left(\frac{(1-\lambda_j L\coth(\lambda_j L))}{\sinh(\lambda_j L)}\right)J_0(\lambda_j R)$$
$$+\sum_{n=1}^{\infty}D_n\left(k_n R\left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2}-1\right)-\frac{2-2\nu}{k_n R}\right) . \tag{68}$$

Similarly, from the boundary condition $\sigma^{\mathrm{h}}_{zz}(r, z = \pm L) = -\sigma^{\mathrm{p}}_{zz}(r, z = \pm L)$ valid for all $r \in [0, \pm R)$, one can deduce that in particular $\sigma^{\mathrm{h}}_{zz}(r = 0, z = L) = -\sigma^{\mathrm{p}}_{zz}(r = 0, z = L)$ must be valid. Then Eq. (51) simplifies to

$$-\hat{c}_0 + \frac{R^2}{4}\hat{c}_2 + \sum_{j=1}^{\infty} \frac{2}{\lambda_j^2 J_0(\zeta_j)}\hat{c}_2 = (6-6\nu)B_0 + (8-4\nu)D_0 + \mathcal{Z}\,, \tag{69}$$

where we abbreviated

$$\mathcal{Z} = \sum_{j=1}^{\infty} B_j \left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right) + \sum_{n=1}^{\infty} (-1)^n D_n \left(2 - k_n R \frac{I_0(k_n R)}{I_1(k_n R)}\right) \frac{1}{I_1(k_n R)}\ . \tag{70}$$

With $\sum_{j=1}^{\infty} \frac{1}{J_0(\zeta_j)\lambda_j^2} = -\frac{R^2}{8}$ (compare Eq. (B.12)) we obtain

$$-\hat{c}_0 = (6-6\nu)B_0 + (8-4\nu)D_0 + \mathcal{Z}\ . \tag{71}$$

Addition of Eqs. (67) and (71) gives

$$-\hat{c}_0 - \hat{c}_1 = 6(B_0 + D_0) + \mathcal{S} \tag{72}$$

with $\mathcal{S} = \mathcal{R} + \mathcal{Z}$ from which we can determine B_0 and D_0. Eq. (72) yields

$$B_0 = -\frac{1}{6}\left(\hat{c}_0 + \hat{c}_1 + \mathcal{S}\right) - D_0\ . \tag{73}$$

and by inserting (73) into (67) we get

$$D_0 = \frac{1}{2(1+\nu)}\left((1-\nu)\hat{c}_1 - \nu\left(\hat{c}_0 + \mathcal{S}\right) + \mathcal{R}\right)\ . \tag{74}$$

The unknowns c and d from the particular solution part can be chosen arbitrarily. Their influence on the homogeneous solution part is restricted to B_0 and D_0 and is compensated in the complete solution.

4.2.6. Summary: Coefficients of Homogeneous Solution For the sake of clearness the equations for the determination of the homogeneous solution part as derived in Eqs. (47), (49), (73), (74), (63) and (65) are summarized:

$$A_j\lambda_j = -B_j\left(2\nu + \lambda_j L \coth(\lambda_j L)\right) \tag{75}$$

$$C_n k_n = -D_n\left((2-2\nu) + k_n R \frac{I_0(k_n R)}{I_1(k_n R)}\right) \tag{76}$$

$$D_n = -\left(k_n R\left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1\right) - \frac{2-2\nu}{k_n R}\right)^{-1} \times \left(\frac{4\hat{c}_2(-1)^n}{k_n^2} + \sum_{m=1}^{\infty} B_m (-1)^n \frac{4\lambda_m k_n^2}{L(k_n^2 + \lambda_m^2)^2} J_0(\lambda_m R)\right) \tag{77}$$

$$B_j = \left(J_0(\lambda_j R)\left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right)\right)^{-1} \times \left(\frac{2}{\lambda_j^2}\hat{c}_2 - \sum_{k=1}^{\infty} (-1)^k D_k k_k \left(\frac{4\lambda_j^2}{R(k_k^2 + \lambda_j^2)^2}\right)\right) \tag{78}$$

$$B_0 = \frac{1}{6}\left(-\hat{c}_0 - \hat{c}_1 - \mathcal{S}\right) - D_0 \tag{79}$$

$$D_0 = \frac{1}{2(1+\nu)}\left((1-\nu)\hat{c}_1 - \nu\left(\hat{c}_0 + \mathcal{S}\right) + \mathcal{R}\right) \tag{80}$$

with

$$\hat{c}_0 = \gamma\left(\frac{4c\nu}{1-\nu} + 2(L^2+d)\right)\ \hat{c}_1 = \gamma\left(\frac{(2\nu-3)}{4(1-\nu)}R^2 + \frac{2(c+\nu d)}{1-\nu}\right)\ \hat{c}_2 = \gamma\frac{2\nu}{1-\nu} \tag{81}$$

and $\mathcal{S} = \mathcal{R} + \mathcal{Z}$.

Now we have determined all unknown coefficients in Equation (44). Then, following Eq. (41) we can directly calculate the homogeneous displacement. By adding the homogeneous and the particular solution we get the total displacement of the cylinder.

4.2.7. Convergence of the Homogeneous Solution The infinite system (77,78) can be approximately solved by reducing it to a finite system, that is, by expanding the sums only to $n = N$ and $j = J$. Then we have a system of $N + J$ equations. By increasing the values of N and J one can improve the accuracy of the solution and find their limits.

This approach is justified, since we can prove that these infinite sums do converge, that means, that the infinite system (77,78) possesses a unique bounded solution. In doing so we have to use the theory of regular infinite systems as has been done by [12] and [21].

In the following we will show first that the equation system (77,78) is a regular infinite system and second that it possesses a unique solution. For the sake of simplicity we introduce the abbreviations

$$P_n = R^2\left(\frac{I_0(k_nR)^2}{I_1(k_nR)^2} - 1\right) - \frac{2-2\nu}{k_n^2} \tag{82}$$

$$Q_j = \frac{L}{\lambda_j}\left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right) \tag{83}$$

$$X_n = D_n(-1)^n\frac{k_n}{R} \tag{84}$$

$$Y_j = -B_j\frac{\lambda_j}{L}J_0(\lambda_j R) \tag{85}$$

and rewrite the system (77,78) as

$$\begin{aligned} X_n - \frac{1}{P_n}\sum_{j=1}^{\infty}\frac{4k_n^2}{(k_n^2+\lambda_j^2)^2}Y_j &= X_n - \sum_{j=1}^{\infty}c_{n,j}Y_j = -\frac{1}{P_n}\frac{4\hat{c}_2}{k_n^2} \\ Y_j - \frac{1}{Q_j}\sum_{n=1}^{\infty}\frac{4\lambda_j^2}{(k_n^2+\lambda_j^2)^2}X_n &= Y_j - \sum_{n=1}^{\infty}d_{j,n}X_n = -\frac{1}{Q_j}\frac{2\hat{c}_2}{\lambda_j^2}\ . \end{aligned} \tag{86}$$

The system (86) can be written in a combined form

$$z_i = \sum_{l=1}^{\infty} e_{i,l}z_l + b_i\,, \qquad i = 1, 2, \ldots, \infty\,, \tag{87}$$

with $z_{2l-1} = X_l$ and $z_{2l} = Y_l$. Thus the vector z alternating contains the terms X_n and Y_j from Equation (86). Then the matrix e must fulfill

$$e_{2m-1,2l-1} = 0 \tag{88}$$

$$e_{2m-1,2l} = c_{m,l} \tag{89}$$

$$e_{2m,2l-1} = d_{m,l} \tag{90}$$

$$e_{2m,2l} = 0 \tag{91}$$

with

$$c_{n,j} := \frac{1}{P_n}\frac{4k_n^2}{(k_n^2+\lambda_j^2)^2} \qquad \text{and} \qquad d_{j,n} := \frac{1}{Q_j}\frac{4\lambda_j^2}{(k_n^2+\lambda_j^2)^2} \; . \tag{92}$$

An infinite system of the form (87) is called *regular*, if in each equation of this system the sum of the norms of the coefficients is smaller than one (see [10])

$$\sum_{l=1}^{\infty} |e_{i,l}| < 1 \qquad (i = 1, 2, \dots) \; . \tag{93}$$

Due to our substitutions we have

$$\sum_{l=1}^{\infty} |e_{i,l}| = \sum_{l=1}^{\infty} |c_{m,l}| \qquad (i = 2m-1) \tag{94}$$

$$\sum_{l=1}^{\infty} |e_{i,l}| = \sum_{l=1}^{\infty} |d_{m,l}| \qquad (i = 2m) \tag{95}$$

so that the condition for regularity reads

$$\sum_{l=1}^{\infty} |c_{m,l}| < 1 \tag{96}$$

$$\sum_{l=1}^{\infty} |d_{m,l}| < 1 \; . \tag{97}$$

These conditions are satisfied for the system (77,78) or (86) because we can calculate using (B.10) and (B.8)

$$\begin{aligned}
\sum_{j=1}^{\infty} c_{n,j} = \frac{1}{P_n}\sum_{j=1}^{\infty}\frac{4k_n^2}{(k_n^2+\lambda_j^2)^2} &= \frac{1}{P_n}R^2\left(\frac{I_0(k_nR)^2}{I_1(k_nR)^2}-1\right) - \frac{4}{k_n^2} \\
&= \frac{1}{P_n}\left(P_n + \frac{2-2\nu}{k_n^2}\right) - \frac{4}{k_n^2} \\
&= 1 + \frac{1}{P_n}\left(\frac{2-2\nu-4}{k_n^2}\right) \\
&= 1 - \Psi_n \qquad \text{with} \qquad \Psi_n = \frac{2+2\nu}{k_n^2 P_n} \\
\sum_{n=1}^{\infty} d_{j,n} = \frac{1}{Q_j}\sum_{n=1}^{\infty}\frac{4\lambda_j^2}{(k_n^2+\lambda_j^2)^2} &= \frac{1}{Q_j}\frac{L}{\lambda_j}\left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} - \frac{2}{\lambda_j L}\right) \\
&= \frac{1}{Q_j}Q_j - \frac{2}{Q_j\lambda_j^2} \\
&= 1 - \frac{2}{Q_j\lambda_j^2} \\
&= 1 - \Phi_j \qquad \text{with} \qquad \Phi_j = \frac{2}{\lambda_j^2 Q_j} \; .
\end{aligned} \tag{98}$$

Obviously, $Q_j > 0$ as the cylinder length L and the radius R are positive and the zeros of the Bessel function of first order $\lambda_j R$ are positive, too. Using properties of the Bessel functions, it can be shown that also $P_n > 0$. Therefore no singularities can occur. Furthermore Ψ and Φ are positive definite series and none of their terms

becomes one for all values of n and j, thus $1 - \Psi_n$ and $1 - \Phi_j$ are always smaller than zero. Therefore, (87) is regular.

The final conclusion that the regular infinite system (77,78) has a unique bounded solution can be inferred from a theorem [10], that a regular infinite system (87) whose free terms b_i fulfill the condition

$$|b_i| \leq K\rho_i \qquad \text{with} \qquad \rho_i := 1 - \sum_{l=1}^{\infty} |e_{i,l}| \tag{99}$$

does have a bounded solution $|z_i| \leq K$, where K is an arbitrary constant.

From (98) its obvious that $\rho_i = 1 - |1 - \Psi_i| = |\Psi_i|$ and $\rho_i = 1 - |1 - \Phi_i| = |\Phi_i|$. Thus the condition $|b_i| \leq K\rho_i$ can be divided into two conditions

$$\left| -\frac{4\hat{c}_2}{P_n k_n^2} \right| \leq K|\Psi_n| = K_1 \left| \frac{2+2\nu}{k_n^2 P_n} \right| \leq K \left| \frac{2+2\nu}{k_n^2 P_n} \right| \tag{100}$$

and

$$\left| -\frac{2\hat{c}_2}{Q_j \lambda_j^2} \right| \leq K|\Phi_j| = K_2 \left| \frac{2}{\lambda_j^2 Q_j} \right| \leq K \left| \frac{2}{\lambda_j^2 Q_j} \right| . \tag{101}$$

Since K_1 and K_2 are arbitrary constants they can be chosen to be

$$K_1 = \frac{4\hat{c}_2}{2+2\nu} \qquad \text{and} \qquad K_2 = \hat{c}_2 \tag{102}$$

which fulfills the above condition. One can choose K to be $K = K_1 = \frac{4\hat{c}_2}{2+2\nu}$, as the Poisson number ν lies between 0 and 0.5 and therefore $K_1 > K_2$. Thus we have proven that the condition for the existence of a bounded solution is fulfilled.

5. Comparison between Analytical and FEM Solution

Having proven that the infinite equation system (77,78) possesses a unique bounded solution, we can now numerically calculate this solution with arbitrary order of accuracy by expanding the infinite series to sufficiently large numbers $N = J$. Although the infinite series converge very quickly, an expansion to higher orders $N = J$ still gives an improvement of accuracy which can be seen in Table 1.

Fig. 2 shows the resulting total displacement field from the analytical solution of Equation (14). The infinite sums in the analytical solution were expanded to $N = J = 1700$.

Now we can compare our analytical solution for our simplified problem with the result obtained with help of a finite element code in order to verify its applicability.

Table 1. Comparison of the analytical solution at point $r = 1 = R$, $z = 2 = L$ for different expansion orders $N = J$ of the infinite sums in equation system (77).

$N = J$	ξ_r	ξ_z
100	$-6.471 \cdot 10^{-15}$	$1.637 \cdot 10^{-13}$
1000	$-6.233 \cdot 10^{-15}$	$1.629 \cdot 10^{-13}$
1700	$-6.218 \cdot 10^{-15}$	$1.628 \cdot 10^{-13}$

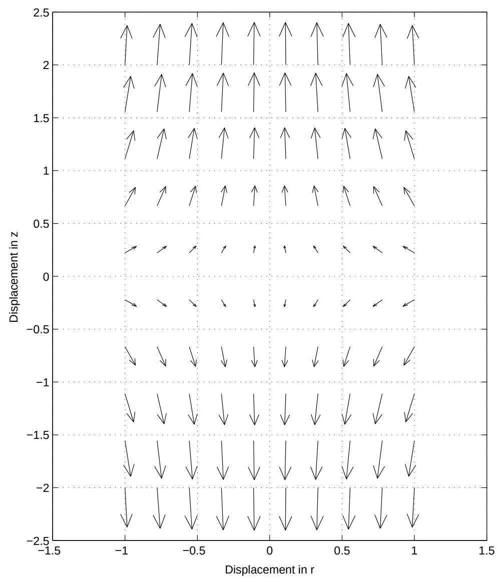

Figure 2. Analytical solution: Displacement field of a cylinder under influence of a spherical tidal gravitational force field. The displacements are plotted over the body coordinates r and z. The cylinder boundaries are at $z = \pm L = \pm 2$ and $r = R = 1$.

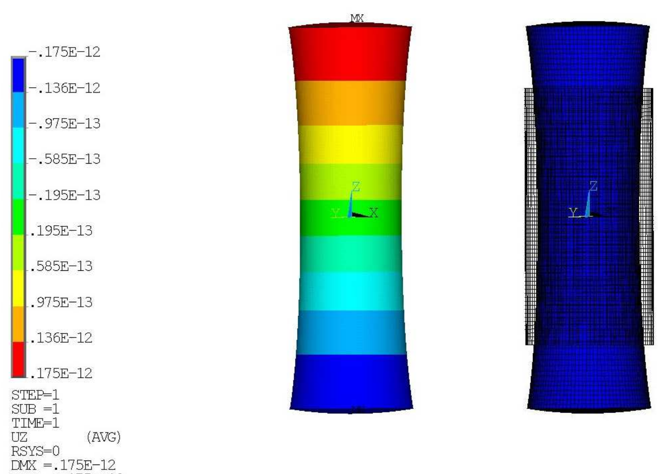

Figure 3. Finite element solution: Deformation of the cylinder under influence of a spherical tidal gravitational force field. The deformation is scaled by a factor of $6 \cdot 10^{13}$. Right: Deformed cylinder shape and original finite element mesh. Left: the scale shows the z displacements.

The finite element analysis was done with the commercial FEM code ANSYS. The cylinder model of radius $R = 1$ and length $L = 2R$ was divided into approximately 110 000 hexahedron elements. Hexahedron elements allow the creation of a structured finite element mesh which ensures a high relative accuracy of the finite element solution. Fig. 3 shows the deformation of the cylinder in the tidal gravitational force

Table 2. Displacements ξ_r and ξ_z in r and z direction; Comparison between analytical and FEM solution. The cylinder boundaries are $r = R = 1$ and $z = \pm L = \pm 2$. In the analytical solution the infinite series were expanded to $N = J = 1700$. The constant $\gamma := -\frac{GM_\oplus}{2r_M^3}\rho = -\frac{3.986004415 \cdot 10^{14}}{2 \cdot 343 \cdot 10^{18}} 2350$ which corresponds to a 7 000 km Earth orbit.

		Analytical		Numerical	
r	z	ξ_r	ξ_z	ξ_r	ξ_z
1	2	$-6.218 \cdot 10^{-15}$	$1.628 \cdot 10^{-13}$	$-6.202 \cdot 10^{-15}$	$1.625 \cdot 10^{-13}$
1	0	$-3.491 \cdot 10^{-14}$	0	$-3.486 \cdot 10^{-14}$	$-1.66 \cdot 10^{-23}$
$0.49507\sqrt{2}$	1	$-2.059 \cdot 10^{-14}$	$1.151 \cdot 10^{-13}$	$-2.062 \cdot 10^{-14}$	$1.150 \cdot 10^{-13}$
$0.11888\sqrt{2}$	1	$-5.208 \cdot 10^{-15}$	$1.183 \cdot 10^{-13}$	$-5.236 \cdot 10^{-15}$	$1.182 \cdot 10^{-13}$

field as result of the FEM calculation.

Table 2 contains a quantitative comparison between the displacements from the analytical and FEM solution for special points of the cylinder. Analytical as well as FEM solution are completely symmetric, i.e. the norms of the displacements of two opposing points of the cylinder are equal. Please note that we have chosen a very fine mesh for the finite element solution in order to get high accuracy.

Obviously, the FEM solution and the analytical solution agree very well. The small differences are based on the nature of the FEM analysis. The elements in which the cylinder is divided cannot be chosen to be infinite small, they are 'finite' and thus the FEM solution is a kind of summation over all elements of finite size instead of an integration where the limit to infinitesimal small element size can be performed. Furthermore during the FEM analysis at least three points must be fixed in order to prevent the cylinder from rigid body rotations. In the current FEM analysis the center of mass as well as four of the next nodes were fixed in order to assure the perfect symmetry of the FEM solution.

6. Summary and Outlook

An analytical solution for the problem of an elastic, isotropic, homogeneous freely–flying cylinder in space under the influence of a tidal gravitational force has been worked out. The motivation for this work was the verification of the use of FEM codes for modeling the deformations of optical resonators in high precision space experiments. It could be shown that the analytical and the FEM solutions are equal within the limits set by the nature of the different solution approaches. The analytical solution itself gives a new application field for the theory of elasticity as so far (according to the knowledge of the authors) no examples with tidal gravitational forces have been calculated.

A further application case of such analytical solutions is the estimation of the noise due to thermoelastic deformations in gravitational wave detectors. In particular with respect to the upcoming LISA (Laser Interferometer Space Antenna) mission [6] these investigations are of great interest. Some groups already dealt with this problem, see [3, 16]. However, the solutions are given for infinite half–spaces only or some approximations have to be included in order to satisfy all boundary conditions. Therefore we are currently working on an exact analytical solution for the estimation of thermoelastic noise in gravitational wave detectors.

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Appendix A. Bessel function relations

For convenience of the reader we collect some equations from [9], [1], [2], [22] used within this article.

The Bessel functions $J_\nu(x)$ of first kind and νth order

$$J_\nu(x) = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!\Gamma(\nu+v+1)} \left(\frac{x}{2}\right)^{\nu+2v} \qquad \nu \in \mathbb{R} \tag{A.1}$$

are solutions of the Bessel differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \ . \tag{A.2}$$

Functions with the argument lx fulfill the differential equation

$$x^2 y'' + xy' + (l^2 x^2 - n^2)y = 0 \ . \tag{A.3}$$

The modified Bessel functions $I_\nu(x)$ of first kind and νth order are defined as

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{v=0}^{\infty} \frac{1}{v!\Gamma(\nu+v+1)} \left(\frac{x}{2}\right)^{\nu+2v} \qquad \nu \in \mathbb{R} \ . \tag{A.4}$$

The orthogonality relations for the Bessel functions can be derived as

$$\frac{R^2}{2} \left(J_{\nu+1}(\zeta_{\nu m})\right)^2 \delta_{mn} = \int_0^R J_\nu(l_{\nu m} r) J_\nu(l_{\nu n} r) r dr \tag{A.5}$$

for $\nu > -1$, $\nu \in \mathbb{R}$ provided that $\zeta_{\nu m} = l_{\nu m} R$ and $\zeta_{\nu n} = l_{\nu n} R$ are the mth respectively the nth zero of J_ν, i.e. $J_\nu(l_{\nu m} R) = 0$ and $J_\nu(l_{\nu n} R) = 0$ and it is shown that they form a complete set, so every arbitrary function $f(r)$ can be represented by a Bessel-Fourier series

$$f(r) = \sum_{m=1}^{\infty} c_{\nu m} J_\nu \left(l_{\nu m} \frac{r}{R}\right) \tag{A.6}$$

for $0 \leq r \leq R, \nu > -1$. The coefficients can be determined via

$$c_{\nu m} = \frac{2}{R^2 [J_{\nu+1}(\zeta_{\nu m})]^2} \int_0^R f(r) J_\nu \left(l_{\nu m} \frac{r}{R}\right) r dr \ . \tag{A.7}$$

Under the condition that $l_{\nu m}$ is related to the mth zero $\zeta_{\nu m}$ of $\frac{\partial}{\partial r} J_\nu(l_{\nu m} r)$ via $\zeta_{\nu m} = l_{\nu m} R$, i.e. $\frac{\partial}{\partial r} J_\nu(l_{\nu m} r)|_{r=R} = 0$, a second orthogonality relation for Bessel functions is

$$\frac{R^2}{2} \left(1 - \frac{\nu^2}{l_{\nu m}^2}\right) \left(J_\nu(\zeta_{\nu m})\right)^2 \delta_{mn} = \int_0^R J_\nu(l_{\nu m} r) J_\nu(l_{\nu n} r) r dr \tag{A.8}$$

for $\nu > -1\nu > -1$, $\nu \in \mathbb{R}$.

Because of this second orthogonality relation one can represent an arbitrary function as expansion of a so-called Dini series

$$f(r) = \sum_{m=1}^{\infty} d_{\nu m} J_\nu \left(l_{\nu m} \frac{r}{R} \right) \tag{A.9}$$

$$d_{\nu m} = \frac{2}{R^2(1-\nu^2/\zeta_m^2)[J_\nu(\zeta_{\nu m})]^2} \int_0^R f(r) J_\nu \left(l_{\nu m} \frac{r}{R} \right) r dr \ . \tag{A.10}$$

Note that in the case of a Dini series for $\nu = 0$ an additional term d_0 has to be added in the series expansion [22]. For Dini series expansions and $\nu = 0$ one has

$$f(r) = d_0 + \sum_{m=1}^{\infty} d_m J_0 \left(l_m \frac{r}{R} \right) \tag{A.11}$$

with

$$d_m = \frac{2}{R^2[J_0(\zeta_m)]^2} \int_0^R f(r) J_0 \left(l_m \frac{r}{R} \right) r dr \tag{A.12}$$

and

$$d_0 = \frac{2}{R^2} \int_0^R f(r) r dr \tag{A.13}$$

where $\zeta_m = l_m R$ is the mth zero of J_1.

Provided that $l_m = \zeta_m / R$, where ζ_m is the mth zero of J_1 and R is the maximum value of r one can derive the special relations

$$\frac{R^2}{2} \left(J_0(\zeta_m) \right)^2 \delta_{mn} = \int_0^R J_1(l_m r) J_1(l_n r) r dr = \frac{R^2}{2} \left(J_2(\zeta_m) \right)^2 \delta_{mn} \ , \tag{A.14}$$

$$\frac{R^2}{2} \left(J_0(\zeta_m) \right)^2 \delta_{mn} = \int_0^R J_0(l_m r) J_0(l_n r) r dr \ , \tag{A.15}$$

$$0 = \int_0^R J_0(l_m r) r dr \ . \tag{A.16}$$

Appendix B. Relations of Fourier and Dini Series

The orthogonality relations for Fourier series are

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin(n'z)\sin(nz) dz = \begin{cases} \pi\delta_{nn'} & n \neq 0 \\ 0 & n = 0 \end{cases} \\
&\int_{-\pi}^{\pi} \cos(n'z)\cos(nz) dz = \begin{cases} \pi\delta_{nn'} & n \neq 0 \\ 2\pi & n = 0 \end{cases} \\
&\int_{-\pi}^{\pi} \sin(n'z)\cos(nz) dz = 0
\end{aligned} \tag{B.1}$$

for integer n and n', $n' \neq 0$.

We also have

$$\int_{-L}^{L} \sin(k_{n'z})\sin(k_n z)dz = \begin{cases} L\delta_{nn'} & n \neq 0 \\ 0 & n = 0 \end{cases}$$
$$\int_{-L}^{L} \cos(k_{n'}z)\cos(k_n z)dz = \begin{cases} L\delta_{nn'} & n \neq 0 \\ 2L & n = 0 \end{cases}$$
$$\int_{-L}^{L} \sin(k_{n'}z)\cos(k_n z)dz = 0 \tag{B.2}$$

for $k_n = n\pi/L$ and $k_{n'} = n'\pi/L$, n, n' are integer numbers, $n' \neq 0$.

As the trigonometric functions form a complete orthogonal set, each function $f(z)$ can be represented by a Fourier series

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n z) + \sum_{n=1}^{\infty} b_n \sin(k_n z) \ . \tag{B.3}$$

The Fourier coefficients are determined via

$$a_n = \frac{1}{L}\int_{-L}^{L} f(z)\cos\left(\frac{n\pi}{L}z\right)dz \,, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(z)\sin\left(\frac{n\pi}{L}z\right)dz \ . \tag{B.4}$$

In the following we collect some Fourier and Dini series expansions used within the article [9], [21], for $k_n = n\pi/L$ and $J_1(\lambda_j R) = 0$, integer n, j:

$$\lambda z\frac{\sinh(\lambda z)}{\sinh(\lambda L)} + (1 - \lambda L\coth(\lambda L))\frac{\cosh(\lambda z)}{\sinh(\lambda L)} = \sum_{n=1}^{\infty}(-1)^n \frac{4\lambda k_n^2}{L(k_n^2+\lambda^2)^2}\cos(k_n z) \tag{B.5}$$

$$r\frac{I_1(kr)}{I_1(kR)} + \left(\frac{2}{k} - R\frac{I_0(kR)}{I_1(kR)}\right)\frac{I_0(kr)}{I_1(kR)} = \sum_{j=1}^{\infty}\frac{4\lambda_j^2}{R(k^2+\lambda_j^2)^2}\frac{J_0(\lambda_j r)}{J_0(\lambda_j R)} \tag{B.6}$$

$$\frac{I_0(kr)}{I_1(kR)} = \frac{2}{kR} + \sum_{j=1}^{\infty}\frac{2k}{R(k^2+\lambda_j^2)}\frac{J_0(\lambda_j r)}{J_0(\lambda_j R)} \tag{B.7}$$

From these equations one can derive the following sums, by setting $z = L$ and $r = R$:

$$\sum_{n=1}^{\infty}\frac{4\lambda^2}{(k_n^2+\lambda^2)^2} = \frac{L}{\lambda}\left(\coth(\lambda L) + \frac{\lambda L}{\sinh^2(\lambda L)} - \frac{2}{\lambda L}\right) \tag{B.8}$$

$$\sum_{j=1}^{\infty}\frac{4\lambda_j^2}{(k^2+\lambda_j^2)^2} = R^2\left(1 - \frac{I_0(kR)^2}{I_1(kR)^2}\right) + \frac{2RI_0(kR)}{kI_1(kR)} \tag{B.9}$$

$$\sum_{j=1}^{\infty}\frac{4k^2}{(k^2+\lambda_j^2)^2} = R^2\left(\frac{I_0(kR)^2}{I_1(kR)^2} - 1\right) - \frac{4}{k^2} \tag{B.10}$$

If $k \to 0$ in Equation (B.9) one gets

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2} = \frac{R^2}{8} \ . \tag{B.11}$$

In particular,

$$\sum_{j=1}^{\infty}\frac{1}{J_0(\lambda_j R)\lambda_j^2} = -\frac{R^2}{8} \ . \tag{B.12}$$

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